1 Notes on empricial Bayes credibility estimation

- r policy holders indexed by i
- X_{ij} is observed loss per unit of exposure for policy holder i, period j.
- $\mathbf{X}_i = (X_{i1}, \ldots, X_{n_i})$ are assumed independent $i = 1, \ldots, r$
- where θ_i is risk parameter for policy holder i, $\theta_1, \ldots, \theta_r$ are assumed independent with density $\pi(\theta)$.
- X_{ij} given θ_i are assumed to be independent $f(x_{ij}|\theta_i)$.
- Exposure vector fo policy holder i is $\mathbf{m}_i = (m_{i1}, \ldots, m_{in_i})$
- m_i is sum of entries in $\mathbf{m}_i$ is total exposure for policyholder i. m is total exposure for all r individuals.
- $\bar{X}_i = \frac{1}{m_i}\sum_{j=1}^{n_i} m_{ij}X_{ij}$ is average loss for individual i.
- $\bar{X} = \frac{1}{m}\sum_{i=1}^{r} m_i\bar{X}_i$ is overall average loss.

2 Bulmann-Straub assumptions/definitions

- $E(X_{ij}|\theta_i) = \mu(\theta_i)$
- $V(X_{ij}|\theta_i) = \frac{v(\theta_i)}{m_{ij}}$
- $\mu = E(\mu(\Theta_i))$
- $v = E(v(\Theta_i))$
- $a = V(\mu(\Theta_i))$
- Credibility premium for next year's losses (per unit of exposure) for individual i is

$$Z_i\bar{X}_i + (1 - Z_i)\mu$$

  - with $Z_i = \frac{m_i}{m_i + k}$ and $k = \frac{v}{a}$.
  - empirical Bayes estimator of credibility premium is

$$\hat{Z}_i\bar{X}_i + (1 - \hat{Z}_i)\hat{\mu}$$

  - with unknowns v, a, mu replaced by esitmates

Example 18.1

- $n_i = n > 1$ for each individual
- $m_{ij} = 1$ for each i, j

In this case, estimators of the model parameters are

- An unbiased estimator is $\hat{\mu} = \bar{X}$, as $\bar{X} = \frac{1}{rn} \Sigma_i \Sigma_j X_{ij}$
- $\hat{v}_i = \frac{1}{n-1} \Sigma_{j=1}^{n} (X_{ij} - \bar{X}_i)^2$
  and $E(v_i) = v$ (Why: $\hat{v}_i$ is a sample variance of i.i.d. observations)
  Combining the r estimates of v, we get an overall estimate
  $\hat{v} = \frac{1}{r} \Sigma_{i=1}^{r} \hat{v}_i$. (Each is unbiased so the average is unbiased.)
- $\hat{a} = \frac{1}{r-1} \Sigma_{i=1}^{r} (\bar{X}_i - \bar{X})^2 - \frac{\hat{v}}{n}$ (follows from $V(\bar{X}_i) = a + v/n$, using conditioning on θ).

Example 18.2

$r = 2$ policy holders with loss histories (3,5,7) and (6,12,9). Goal is to estimate Buhlmann credibility premiums for each.

$\bar{X}_1 = 5$, $\bar{X}_2 = 9$, so $\hat{\mu} = \bar{X} = .5(5 + 9) = 7$.

$\hat{v}_1$ is sample variance of (3,5,7), or $8/2 = 4$. $\hat{v}_2 = 18/2 = 9$.

$\hat{v}$ is the average of the $\hat{v}_i$, so $(4+9)/2 = 13/2$.

$\hat{a}$ is the sample variance of the $\bar{X}_i$ minus $\hat{v}/n$, so 8-(13/2)/3 = 35/6.

$\hat{k} = \hat{v}/\hat{a} = (13/2)/(35/6) = 39/35$, giving $\hat{Z} = \frac{3}{3+39/35} = 105/144 = 35/48$.

Estimated credibility premiums are, for i=1,2,

$$\hat{Z}\bar{X}_i + (1 - \hat{Z})\hat{\mu}$$

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Example 18.3

- Suppose $X_1, \ldots, X_n$ independent with common mean μ and variance $V(X_j) = \beta + \alpha/m_j$. [This seems like an unusual choice, but a special cases will be used below.]
- Both $\bar{X} = \frac{1}{m}\Sigma_{j=1}^n m_j X_j$ and $\hat{\mu}_1 = \frac{1}{n}\Sigma_{j=1}^n X_j$ are easily shown to be unbiased for μ.

  It is typical that there are many unbiased estimators. Better ones have smaller variances. It is shown that each of these estimators can have the smaller variance depending on the choice of α and β, one is not always better than the other.
- $E(\Sigma_{j=1}^n m_j(X_j - \bar{X})^2) = E(\Sigma_{j=1}^n m_j((X_j - \mu) + (\mu - \bar{X}))^2)$, leading to

$$E(\sum_{j=1}^n m_j(X_j - \bar{X})^2) = \beta(m - \frac{1}{m}\sum_{j=1}^n m_j^2) + \alpha(n-1) \qquad (1)$$

unbiased estimate of v in Bulmann-Straub model

For $i = 1, \ldots, r$, define $\hat{v}_i = \frac{1}{n_i - 1} \Sigma_{j=1}^{n} m_j (X_{ij} - \bar{X}_i)^2$. Using (1) with $\beta = 0$ and $\alpha = v(\Theta_i)$, leads to

$$E(\hat{v}_i) = E(E(\hat{v}_i | \Theta_i)) = v$$

We have r unbiased estimates of v which are combined to get $\hat{v} = \Sigma_{i=1}^{r} w_i \hat{v}_i$. This is also unbiased for v provided the weights sum to 1. Choosing w_i proportional to $n_i - 1$ gives an unbiased estimator of v as:

$$\hat{v} = \frac{\Sigma_i \Sigma_j m_{ij} (X_{ij} - \bar{X}_i)^2}{\Sigma_i (n_i - 1)}$$

unbiased estimate of a in Buhlmann-Straub model

For fixed i, $X_{i1}, \ldots, X_{n_i}$ are assumed conditionally independent with variances $\frac{v(\Theta_i)}{m_{ij}}$, so that $V(\bar{X}_i) = v(\Theta_i)/m_i$.

Then calculate $V(\bar{X}_i) = E(V(\bar{X}_i | \Theta_i)) + V(E(\bar{X}_i | \Theta_i))$ to find

$$V(\bar{X}_i) = a + \frac{v}{m_i}$$

This means that $\bar{X}_1, \ldots, \bar{X}_r$ are independent with common mean μ and variances $V(\bar{X}_i)$. Plugging into (1) from example 18.3, with $\beta = a$ and $\alpha = v$, it follows that

$$E(\sum_{i=1}^{r} m_j (\bar{X}_i - \bar{X})^2) = a(m - \frac{1}{m} \sum_{i=1}^{r} m_i^2) + v(r - 1)$$

Replace the expectation by the random variable $\Sigma_{i=1}^{r} m_j (\bar{X}_i - \bar{X})^2$, replace v by $\hat{v}$, and solving for a gives the unbiased estimator

$$\hat{a} = \left(m - m^{-1} \sum_{i=1}^{r} m_i^2 \right)^{-1} \left[\sum_{i=1}^{r} m_i (\bar{X}_i - \bar{X})^2 - \hat{v}(r - 1) \right]$$

Example 18.4

Two policy holders.

Policy holder 1. No policy in year 1. In year 2 total claims are 10000 with 50 insured. In year 3 13000 total claims with 60 insured. In year 4 there are 75 insured.

Policy holder 2. In year 1, 18000 total claim with 100 insured. Year 2, 21000 total claim with 110 insured. Year 3, 17000 total claim with 105 insured. In year 4 there are 90 insured.

What are the estimated credibility premiums for each policy holder in year 4?

For policy holder 1

- $n_1 = 2$, $n_2 = 3$.
- $m_{11} = 50$, $X_{11} = 10000/50 = 200$, $m_{12} = 60$, $X_{12} = 13000/60 = 216.67$,
  $m_1 = m_{11} + m_{12} = 110$
  $\bar{X}_1 = (10000 + 13000)/110 = 209.09$

For policy holder 2
$m_2 = 315$, $\bar{X}_2 = 177.78$

- $m = m_1 + m_2 = 425$, $\hat{\mu} = \bar{X} = 79000/425 = 185.88$.
- $$\hat{v} = (50(200 - 209.09)^2 + \ldots + (161.9 - 177.78)^2)/((2-1) + (3-1)) = 17837.87$$
- $$\hat{a} = \frac{110(209.09 - 185.88)^2 + 315(177.78 - 185.88)^2 - 17837.87}{425 - (110^2 + 315)^2/425} = 380.76$$
- $$\hat{k} = \hat{v}/\hat{a} = 46.85$$
- $\hat{Z}_1 = 110/(110 + 46.85)$, $\hat{Z}_2 = 315/(315 + 46.85)$
- For policy holder 1, estimated credibility premium per individual is $\hat{Z}_1\bar{X}_1 + (1 - \hat{Z}_1)\hat{\mu} = 202.13$, so for the group this is multiplied by 75. with a similar calculation for policy holder 2.