STAT 3340 Assignment 3, Fall 2024 - Solutions

(out of 35 points)

1. A regression analysis is to be carried out for the model $y = \beta_0 + \beta_1 X_1 + \epsilon$.

There are six observations, with $y^{'} = (3, 2, 1, 1, 2, 0)$ and

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

- 1a) Find $(X'X)^{-1}$ (3 points)

$$X'X = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$(X'X)^{-1} = \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/12 \end{bmatrix}$$

- 1b) Find the (6,6) element of $X(X'X)^{-1}X'$. (3 points)

This is the 6'th row of X times $(X'X)^{-1}$ times the 6'th column of X', or

$$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{bmatrix} 1/6 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/12 \end{bmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/6 & -1/6 & 1/12 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 5/12$$

- 1c) Find $X'y$ (1 point)

$X'y = (9, 3, -3)^T$

- 1d) find $\hat{\beta}$ (1 point)

$\hat{\beta} = (9/6, 3/6, -3/12)^T)$.

- 2. Suppose that X and Y are both have mean 0 and variance 1, and that the covariance between X and Y is -.5.

  - 2a) What are the mean and variance of X+Y? (1 point for mean, 1 point for variance)

$E(X+Y) = E(X) + E(Y) = 0 + 0 = 0$

$V(X+Y) = V(X) + V(Y) + 2Cov(X,Y) = 1 + 1 + 2(-.5) = 1$

  - 2b) What are the mean and variance of $X - 2Y$? (1 point for mean, 2 points for variance)

$E(X-2Y) = E(X) - 2E(Y) = 0$

$V(X-2Y) = COV(X-2Y, X-2Y) = COV(X,X) + COV(X,-2Y) + COV(-2Y,X) + COV(-2Y,-2Y)$

$$= V(X) - 4COV(X,Y) + 4V(Y) = 1 - 4(-.5) + 4 = 7$$

  - 2c) What is the covariance between X+Y and 2X-3Y? (2 points)

$COV(X+Y, 2X-3Y) = 2COV(X,X) - 3COV(X,Y) + 2COV(Y,X) - 3COV(Y,Y)$

$= 2V(X) - COV(X,Y) - 3V(Y) = 2 + .5 - 3 = -.5$

- 3. For the data $y = (1, 2, 4)$ and $x = (0, 1, 2)$, in order to fit the model $y = \beta_0 + \beta_1 x + \epsilon$ using matrix calculations, what is the appropriate matrix $\mathbf{X}$? (2 points)

$$\mathbf{X} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$$

- 4. Suppose X, Y and Z are random variables with means $\mu_X = 1$, $\mu_Y = 2$ and $\mu_Z = 3$, variances $\sigma_X^2 = 9$, $\sigma_Y^2 = 4$ and $\sigma_Z^2 = 1$, and covariances $Cov(X, Y) = 3$, $Cov(Y, Z) = 1$ and $Cov(X, Z) = 1.5$. Let $U = X - Y + 2Z$ and $V = X + 2Y - 2Z$.

- 4a) What is $E(U)$? (2 points)

$$E(X - Y + 2Z) = E(X) - E(Y) + 2E(Z) = 1 - 2 + 2(3) = 5$$

- 4b) What is $Var(U)$? (2 points)

$$V(X) + (-1)^2 V(Y) + 2^2 V(Z) + 2(1)(-1)COV(X, Y) + 2(1)(2)COV(X, Z) + 2(-1)(2)COV(Y, Z)$$
$$= 9 + 4 + 4 - 2(3) + 4(1.5) - 4(1) = 13$$

- 4c) What is $Cov(U, V)$? (3 points)

$$\begin{aligned}
&COV(X, X) + COV(X, 2Y) + COV(X, -2Z) \\
&+ COV(-Y, X) + COV(-Y, 2Y) + COV(-Y, -2Z) \\
&+ COV(2Z, X) + COV(2Z, 2Y) + COV(2Z, -2Z) \\
\\
&= V(X) + 2COV(X, Y) - 2COV(X, Z) \\
&- COV(Y, X) - 2V(Y) + 2COV(Y, Z) \\
&+ 2COV(Z, X) + 4COV(Z, Y) - 4V(Z) \\
\\
&= 9 + 2(3) - 2(1.5) - 3 - 2(4) + 2(1) + 2(1.5) + 4(1) - 4(1) = 6
\end{aligned}$$

- 5. Suppose that Y is a random vector with mean vector $(3, 0, 1)'$, and covariance matrix

$$\begin{bmatrix} 1\ 0\ 0 \\ 0\ 2\ 0 \\ 0\ 0\ 3 \end{bmatrix}$$

`Let`

$\mathbf{b} = (-3, 0, 1)'$ and

$$\mathbf{A} = \begin{bmatrix} 1\ 0\ 3 \\ 0\ 0\ 2 \\ 3\ 2\ 4 \end{bmatrix}$$

- 5a) What is the mean of $\mathbf{AY} + \mathbf{b}$? (2 points)

$$\mathbf{A}E(Y) + \mathbf{b} = \begin{bmatrix} 1\ 0\ 3 \\ 0\ 0\ 2 \\ 3\ 2\ 4 \end{bmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 14 \end{pmatrix}$$

- 5b) What is the covariance matrix of $\mathbf{AY} + \mathbf{b}$? (3 points)

$$\mathbf{A}\Sigma\mathbf{A}^T = \begin{bmatrix} 1\ 0\ 3 \\ 0\ 0\ 2 \\ 3\ 2\ 4 \end{bmatrix} \begin{bmatrix} 1\ 0\ 0 \\ 0\ 2\ 0 \\ 0\ 0\ 3 \end{bmatrix} \begin{bmatrix} 1\ 0\ 3 \\ 0\ 0\ 2 \\ 3\ 2\ 4 \end{bmatrix} = \begin{bmatrix} 1\ 0\ 9 \\ 0\ 0\ 6 \\ 3\ 4\ 12 \end{bmatrix} \begin{bmatrix} 1\ 0\ 3 \\ 0\ 0\ 2 \\ 3\ 2\ 4 \end{bmatrix} = \begin{bmatrix} 28\ 18\ 39 \\ 18\ 12\ 24 \\ 39\ 24\ 65 \end{bmatrix}$$

- 5c) What is the mean of $\mathbf{Y}'\mathbf{AY}$? (3 points)

$$\mu^T\mathbf{A}\mu + tr(\Sigma\mathbf{A}) = \begin{pmatrix} 3\ 0\ 1 \end{pmatrix} \begin{bmatrix} 1\ 0\ 3 \\ 0\ 0\ 2 \\ 3\ 2\ 4 \end{bmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + tr\left(\begin{bmatrix} 1\ 0\ 3 \\ 0\ 0\ 4 \\ 9\ 6\ 12 \end{bmatrix}\right) = 31 + 13 = 44$$

- 6. An extension of the usual multiple regression model is the 'mixed effects' model

$$y = X\beta + Zu + \epsilon$$

  The mixed effects model assumes that
  - ϵ is a random vector with mean vector 0 and covariance matrix $\sigma^2 I$
  - u is a random vector with mean vector 0 and covariance matrix $\tau^2 I$
  - u and ϵ are independent of one another
  - X, Z are matrices of known constants, and β is a vector of constants.

  Find $Cov(y, u)$, the covariance between y and u. (3 points)

$$COV(y, u) = COV(X\beta + Zu + \epsilon, u) = COV(X\beta, u) + COV(Zu, u) + COV(\epsilon, u)$$

$$= 0 + ZCov(u, u)I + 0 = Z\tau^2 II = \tau^2 Z$$