Confidence intervals (CI's) I

- There are two equivalent ways of defining confidence intervals for a parameter θ.

First Method I

- The first defines a confidence interval to be the realization of a random interval which contains the true value of the parameter with a high probability.
- Denote the endpoints of the CI as a function of the data $\mathbf{y}$ as $L(\mathbf{y})$ and $U(\mathbf{y})$.
- They are constructed so there is a high probability that θ is contained in the interval, i.e.

$$P(L(\mathbf{y}) \ \leq \ \theta \ \leq \ U(\mathbf{y})) \ = \ 1 - \alpha$$

  where α is chosen to be small, usually .05 or .01, so that $1 - \alpha$ is large.

- Once the data is collected, the functions L and U are evaluated giving the interval $(L(\mathbf{y}), U(\mathbf{y}))$.

First Method II

- There is now nothing random about the interval, so it is not possible to assign probability to the interval, causing us to use the word ‘confidence’.
- Because of the way it was constructed, such an interval will contain the true, but unknown value with probability $1 - \alpha$. That is $100(1 - \alpha)\%$ of such intervals will contain the unknown parameter.
- With the data we have, we can not determine whether the true value is inside or not.

Second Method I

- In most cases, the endpoints are obtained together from something called a pivotal function.
- This pivotal function is often a test statistic whose distribution does not depend on the unknown parameter of interest. This gives the second approach to defining a confidence interval.
- In this context, a confidence interval is the set of values for θ_0 which are not statistically significant at level α (i.e. $P > \alpha$) when used in the null hypothesis of a hypothesis test, that is, when testing $H_0 : \boldsymbol{\theta} = \theta_0$, against the two sided alternative $H_0 : \boldsymbol{\theta} \neq \theta_0$

- For example, the quantity

$$T = \frac{\hat{\theta} - \theta}{s(\hat{\theta})}$$

frequently has a t distribution, and is used in hypothesis testing.

- The pivoting argument begins by noting that

$$P(-t_{\alpha/2} \;\leq\; \frac{\hat{\theta} - \theta}{s(\hat{\theta})} \;\leq\; t_{\alpha/2}) \;=\; 1 - \alpha$$

where $t_{\alpha/2}$ is the upper quantile, cutting off $\alpha/2$ in the right tail.

Example: t-intervals II

- If θ is the value assumed true in the null hypothesis, the values of $\hat{\theta}$ satisfying the two inequalities above lead to p-value greater than α, so non-significant when testing at level α.
- Rearranging, by multiplying all three terms by the standard error, gives

$$P(-t_{\alpha/2}s(\hat{\theta}) \ \leq \ \hat{\theta} \ - \ \theta \ \leq \ t_{\alpha/2}s(\hat{\theta})) \ = \ 1 - \alpha.$$

- Subtracting $\hat{\theta}$ from all three terms, gives

$$P(-\hat{\theta} - t_{\alpha/2}s(\hat{\theta}) \leq -\theta \leq -\hat{\theta} + t_{\alpha/2}s(\hat{\theta})) = 1 - \alpha$$

Example: t-intervals III

- Then multiplying by -1, which changes the direction of the inequalities, gives

$$P(\hat{\theta} - t_{\alpha/2}s(\hat{\theta}) \leq \theta \leq \hat{\theta} + t_{\alpha/2}s(\hat{\theta}))$$

$= 1- \alpha$

- In this case the functions $L(\mathbf{y})$ and $U(\mathbf{y})$ are $\hat{\theta} \; - \; t_{\alpha/2}s(\hat{\theta})$ and $\hat{\theta} \; + \; t_{\alpha/2}s(\hat{\theta})$
- The degrees of freedom associated with the t-distribution will depend on the particular context.

Confidence intervals for simple linear regression I

- Confidence intervals for the intercept and slope in a simple linear regression are constructed using

$$\hat{\beta}_i \pm t_{\alpha/2,n-2} se(\hat{\beta}_i)$$

for $i = 0$ or $i = 1$.

- The degrees of freedom for the t distribution are the same as the degrees of freedom associated with MS_{Res}, also known as MSE or MS_{error}.
- A confidence interval for the mean of Y at $x = x_0$ is

$$\hat{\mu}_{x_0} \pm t_{\alpha/2,n-2} se(\hat{\mu}_{x_0}).$$