1 Random vectors

- $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_m)^T$ is an m dimensional collection of random variables.
- it is referred to as an m dimensional random vector.
- the transpose (^{T} or $'$) means that we focus on column vectors.

2 Means and covariances of random vectors

- Means are calculated componentwise.

$$E(\boldsymbol{Y}) = \boldsymbol{\mu} = (\mu_1, \ldots, \mu_m)^T$$

  where $\mu_j = E(\boldsymbol{Y}_j)$.

- The variances and covariances of the elements of $\boldsymbol{Y}$ are stored in an $m \times m$ dimensional covariance matrix often denoted by $Var(\boldsymbol{Y})$ or $Cov(\boldsymbol{Y})$. This has j'th diagonal element equal to $V(Y_j) = \sigma_j^2$, and with the element in the j'th row, l'th column being the covariance between Y_j and Y_l, denoted by $COV(Y_j, Y_l)$ or σ_{jl}.
- In terms of expected values, the covariance matrix is given by

$$\begin{aligned} Var(\boldsymbol{Y}) &= E(\boldsymbol{Y} - \boldsymbol{\mu})(\boldsymbol{Y} - \boldsymbol{\mu})^T \\ &= E(\boldsymbol{Y}\boldsymbol{Y}^T) - \boldsymbol{\mu}\boldsymbol{\mu}^T. \end{aligned}$$

3 Rules

- If $\boldsymbol{A}$ is a matrix and $\boldsymbol{c}$ a vector of scalars, then

$$E(\boldsymbol{A}\boldsymbol{Y} + \boldsymbol{c}) = \boldsymbol{A}\boldsymbol{\mu} + \boldsymbol{c}$$

$$Var(\boldsymbol{A}\boldsymbol{Y} + \boldsymbol{c}) = \boldsymbol{A}Var(\boldsymbol{Y})\boldsymbol{A}^T = \boldsymbol{A}\boldsymbol{V}\boldsymbol{A}^T.$$

- Suppose $\boldsymbol{X}$ is a random vector with mean $\boldsymbol{\mu}_X$ and covariance matrix $\boldsymbol{V}_X$ and $\boldsymbol{Y}$ is a random vector with mean $\boldsymbol{\mu}_Y$ and covariance matrix $\boldsymbol{V}_Y$. Then

$$\begin{aligned} Cov(\boldsymbol{X}, \boldsymbol{Y}) &= E[(\boldsymbol{X} - \boldsymbol{\mu}_X)(\boldsymbol{Y} - \boldsymbol{\mu}_Y)^T] \\ &= E(\boldsymbol{X}\boldsymbol{Y}^T) - \boldsymbol{\mu}_X\boldsymbol{\mu}_Y^T \end{aligned}$$

- Suppose $Cov(\boldsymbol{X}, \boldsymbol{Y}) = \boldsymbol{C}$. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be non random matrices and $\boldsymbol{c}$ and $\boldsymbol{d}$ non random vectors. Then

$$Cov(\boldsymbol{A}\boldsymbol{X} + \boldsymbol{c}, \boldsymbol{B}\boldsymbol{Y} + \boldsymbol{d}) = \boldsymbol{A}\boldsymbol{C}\boldsymbol{B}^T.$$

- If A is a symmetric matrix, the expectation of the *quadratic form* $\boldsymbol{Y}^T\boldsymbol{A}\boldsymbol{Y}$ is

$$E(\boldsymbol{Y}^T\boldsymbol{A}\boldsymbol{Y}) = \boldsymbol{\mu}^T\boldsymbol{A}\boldsymbol{\mu} + tr(\boldsymbol{V}\boldsymbol{A})$$

4 examples

Suppose that $\boldsymbol{X} = (X_1, X_2, X_3)'$ is a random vector having mean vector $(0, 0, 1)^T$, and covariance matrix

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 3 & 7 \end{bmatrix}$$

Let $\boldsymbol{b} = (-1, 2, 1)^T$ and

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

1. What is $\boldsymbol{AX} + \boldsymbol{b}$ in terms of the elements of $\boldsymbol{X}$, $\boldsymbol{A}$ and $\boldsymbol{b}$?
2. What is the mean of $\boldsymbol{AX} + \boldsymbol{b}$?
3. What is the covariance matrix of $\boldsymbol{AX} + \boldsymbol{b}$?
4. Where

   $$\boldsymbol{M} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & -3 & -1 \\ 2 & -1 & -1 \end{bmatrix}$$

   What is the mean of $\boldsymbol{X}^T\boldsymbol{MX} + 17$?
5. Suppose that $\boldsymbol{Y}$ has mean vector $(1, 2, 3)^T$, and covariance matrix

   $$\begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 3 & 16 \end{bmatrix}$$

   Let $\boldsymbol{c} = (1, 0, 0)^T$ and

   $$\boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ -1 & -2 & -3 \end{bmatrix}$$

   and suppose that the covariance between $\boldsymbol{X}$ and $\boldsymbol{Y}$ is

   $$Cov(\boldsymbol{X}, \boldsymbol{Y}) = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ -1 & 3 & 4 \end{bmatrix}$$

   (a) What is the mean of $\boldsymbol{AX} + \boldsymbol{BY} + \boldsymbol{b} - \boldsymbol{c}$?
   (b) What is the covariance of X_1 with Y_2?
   (c) What is the covariance matrix of $\boldsymbol{AX} + \boldsymbol{b}$ with $\boldsymbol{BY} - \boldsymbol{c}$?

Let μ and Σ denote the mean and covariance matrix of X. The following gives the solutions to the problems above apart from the matrix arithmetic. You can check your calculations by running the R code on the next page.

1. $AX + b = (X_1 - X_2 + 2X_3, 3X_1 - 3X_2 - X_3, -X_1 + X_3)^T$
2. $E(AX + b) = AE(X) + b$
3. $V(AX+b) = Cov(AX+b, AX+b) = Cov(AX, AX) = ACov(X, X)A^T = A\Sigma A^T$
4. $E(X^T MX + 17) = 17 + E(X^T MX) = 17+\mu^T M\mu$ + tr(M Σ)
5. (a) $E(AX + BY + b - c) = AE(X) + BE(Y) + b - c$
   (b) this is the (1,2) element of Cov(X,Y), which is equal to 0
   (c) $Cov(AX + b, BY - c) = Cov(AX, BY) = ACov(X, Y)B^T$

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ux=c(0,0,1)
Sx=matrix(c(1,0,-1,0,2,3,-1,3,7),byrow=T,ncol=3)
Sx

b=c(-1,2,1)
A=matrix(c(1,-1,2,3,-3,-1,-1,0,1),byrow=T,ncol=3)

A%*%ux+b   #mean of AX+b
A%*%Sx%*%t(A) #covariance of AX+b

M=matrix(c(1,-1,2,-1,-3,-1,2,-1,-1),byrow=T,ncol=3)
Sx%*%M # SxM
sum(diag(Sx%*%M)) # trace of SxM

#mean of X'MX+17
t(ux)%*%M%*%ux+sum(diag(Sx%*%M))+17 #requres M to be symmetric

c=c(1,0,0)
B=matrix(c(1,2,3,0,1,0,-1,-2,-3),byrow=T,ncol=3)
CovXY=matrix(c(1,0,-1,2,1,-1,-1,3,4),byrow=T,ncol=3)
uy=c(1,2,3)

CovXY[1,2] #covariance of X1 with Y2 is 1,2 element of CovXY

A%*%ux+B%*%uy+b-c  #mean of AX+BY+b-c
A%*%CovXY%*%t(B) #COV(AX+b,BY+c)
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