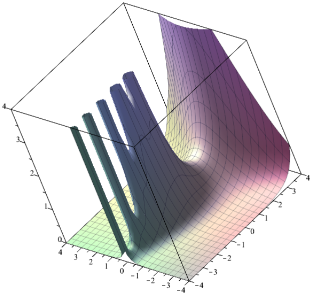
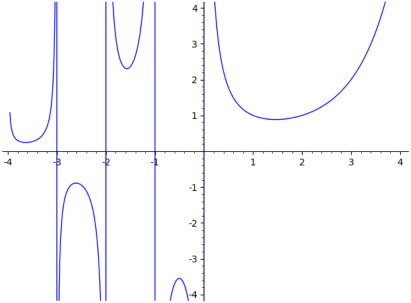
Gamma Function

January 13th

"Poster"





- Maple:

```
plot3d(abs(GAMMA(x + y*I)),x=-4..4,y=-4..4,view=0..4)
```

- SageMath:

```
var('x'); plot(gamma(x),(x,-4,4),ymin=-4,ymax=4)
```

Extension

$$\boxed{f : U \subsetneq A \longrightarrow B} \rightsquigarrow \boxed{F : A \longrightarrow B'(\supset B) \ \text{s.t.}\ F|_U = f}.$$

Example

Let $U = \mathbb{Z}$, $A = \mathbb{R}$, and $f(n) = 0$, for all $n \in \mathbb{Z}$.

$$\begin{cases} F_1(x) & = 0 \\ F_2(x) & = \sin(\pi x) \end{cases} \Longrightarrow F_1|_{\mathbb{Z}} = f = F_2|_{\mathbb{Z}}.$$

- Uniqueness
- Choice

Factorial

For $n \in \mathbb{N}$, $n! = n \cdot (n-1) \cdots 1$. Find $\Gamma : \mathbb{C} \longrightarrow \mathbb{C}$ that extends $n!$.

Definition

The Pochhammer symbol: $(a)_n := a(a+1)\cdots(a+n-1)$.

$$n! = \frac{(2n)!}{(n+1)_n} = \frac{(n+n)!}{(n+1)_n} \Rightarrow m! = \frac{(m+n)!}{(m+1)_n} \quad \text{and} \quad n! = \frac{(m+n)!}{(n+1)_m}$$

$$m! = \frac{n! \cdot (n+1)_m}{(m+1)_n} = \frac{n! \cdot n^m}{(m+1)_n} \cdot \frac{(n+1)_m}{n^m} \qquad \boxed{\lim_{n\to\infty} \frac{(n+1)\cdots(n+m)}{n^m} = 1}$$

$$x! := \lim_{n\to\infty} \frac{n! \cdot n^x}{(x+1)_n}.$$

Remarks

$$x! := \lim_{n\to\infty} \frac{n!\cdot n^x}{(x+1)_n} \left(= \lim_{n\to\infty} \frac{n!\cdot n^x}{(x+1)_n} \cdot \frac{(n+1)^x}{n^x} \right).$$

- x cannot be negative integers. If $x = -m$, then $(x+1)_n = 0$, for $n \geq m$.
- Convergence.

$$\frac{n!\cdot n^x}{(x+1)_n} = \frac{n!\cdot n^x}{(x+1)_n} \cdot \frac{(n+1)^x}{(n+1)^x} = \left(\frac{n}{n+1}\right)^x \prod_{j=1}^{n} \left(1+\frac{x}{j}\right)^{-1} \left(1+\frac{1}{j}\right)^x$$

$$\left(1+\frac{x}{j}\right)^{-1} \left(1+\frac{1}{j}\right)^x = 1 + \frac{x(x-1)}{2j^2} + O\left(\frac{1}{j^3}\right).$$

- Shift

$$\Pi(x) := \lim_{n\to\infty} \frac{n!\cdot n^x}{(x+1)_n} \qquad \Gamma(x) = \Pi(x-1)$$

Gamma function

Definition

For $x \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$,

$$\Gamma(x) := \lim_{k \to \infty} \frac{k! k^{x-1}}{(x)_k}.$$

- $\Gamma(n) = (n-1)!$
- $\frac{k! k^{x}}{(x+1)_k} = \frac{k \cdot k! k^{x-1}}{\frac{(x+k)}{x}(x)_k} \implies \Gamma(x+1) = x\Gamma(x)$
- $\Gamma(1) = 1$, since $(1)_k = k!$

Theorem (Thm. 1.2.2)

Recall $\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)$. *Then,*

$$\frac{1}{\Gamma(x)} = \lim_{n \to \infty} \frac{x(x+1)\cdots(x+n-1)}{n! n^{x-1}} = \cdots = x e^{\gamma x} \prod_{n=1}^{\infty} \left(\left(1 + \frac{x}{n}\right) e^{-\frac{x}{n}}\right).$$

Remark. Weierstrass factorization theorem

Infinite product expression $\frac{1}{\Gamma(x)} = xe^{\gamma x}\prod_{n=1}^{\infty}\left(\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}}\right)$

$$-\log\Gamma(x) = \log x + \gamma x + \sum_{n=1}^{\infty}\log\left(1+\frac{x}{n}\right) - \frac{x}{n} \Rightarrow -\frac{\Gamma'(x)}{\Gamma(x)} = \gamma + \sum_{k=1}^{\infty}\left(\frac{1}{x+k-1} - \frac{1}{k}\right)$$

- $\Gamma'(1) = -\gamma$
- $\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(1)}{\Gamma(1)} = -\sum_{k=0}^{\infty}\left(\frac{1}{x+k} - \frac{1}{1+k}\right)$
- $\frac{\mathrm{d}^2 \log\Gamma(x)}{\mathrm{d}x^2} = \sum_{k=0}^{\infty}\frac{1}{(x+k)^2} \implies \log\Gamma(x)$ is convex for $x > 0$.

Theorem (Bohr-Mollerup Theorem (§1.9))

If f is a positive function on $x > 0$ such that

1. $f(1) = 1$
2. $f(x+1) = xf(x)$
3. *f is logarithmically convex (i.e., $\log f$ is convex)*

Then, $f(x) = \Gamma(x)$ for $x > 0$.

Bohr-Mollerup theorem

Convex: if f is convex in (a, b), for any x, y, z such that $a < x < y < z < b$

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}$$

Since f is log-convex, consider intervals $[n, n+1]$, $[n+1, n+1+x]$, and $[n+1, n+2]$:

$$\log \frac{f(n+1)}{f(n)} \leq \frac{1}{x} \log \frac{f(n+1+x)}{f(n+1)} \leq \log \frac{f(n+2)}{f(n+1)}$$

By $f(1) = 1$ and $f(x+1) = xf(x)$, the inequalities yield

$$0 \leq \log \frac{(x)_{n+1}}{n!n^x} + \log f(x) \leq x \log \left(1 + \frac{1}{n}\right)$$

Now, let $n \to \infty$,

$$f(x) = \lim_{n\to\infty} \frac{n!n^x}{(x)_{n+1}} = \lim_{n\to\infty} \frac{n!n^{x-1}}{(x)_n} \cdot \frac{n}{x+n} = \Gamma(x).$$

Integral Representation

Definition

For $Re(x) > 0$ and $Re(y) > 0$, the beta integral/function is defined by

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t.$$

Theorem

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Corollary

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t \qquad Re(x) > 0.$$

- $\Gamma(x)=\int_0^1 t^{x-1}e^{-t}\mathrm{d}t+\int_1^\infty t^{x-1}e^{-t}\mathrm{d}t=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+x)n!}+\int_1^\infty t^{x-1}e^{-t}\mathrm{d}t$

$$Res(\Gamma,-n)=(-1)^n/n!,\qquad n=0,1,2,\dots$$

- $\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\overset{t=\sin^2\theta}{=\!=\!=\!=}2\int_0^{\frac{\pi}{2}}\sin^{2x-1}\theta\cos^{2y-1}\mathrm{d}\theta\Rightarrow\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

Theorem (Euler's Reflection Formula)

$$\Gamma(x)\Gamma(1-x)=\pi/\sin(\pi x)$$

Proof.

$\Gamma(x)\Gamma(1-x)=B(x,1-x)\overset{t=\frac{s}{s+1}}{=\!=\!=\!=}\int_0^\infty\frac{s^{x-1}}{1+s}\mathrm{d}s$ + Contour Integral. □

Remark (Gauss's Multiplication Formual for $\Gamma(ma)$)

For $m\in\mathbb{N}$, $\Gamma(ma)(2\pi)^{\frac{m-1}{2}}=m^{ma-\frac{1}{2}}\Gamma(a)\Gamma\left(a+\frac{1}{m}\right)\cdots\Gamma\left(a+\frac{m-1}{m}\right)$.

The following formulas can be derived from Euler's reflection formula and the infinite product expression of $1/\Gamma(x)$.

$$\sin(\pi x) = \pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right)$$

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n}\right) = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{x-k}$$

$$\frac{\pi}{\sin(\pi x)} = \frac{1}{x} + 2x \sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 - n^2} = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{(-1)^k}{x-k}$$

$$\pi \tan(\pi x) = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{k + \frac{1}{2} - x}$$

$$\pi \sec(\pi x) = \lim_{n\to\infty} \sum_{k=-n}^{n} \frac{(-1)^k}{k + x + \frac{1}{2}}$$

$$\frac{\pi^2}{\sin^2(\pi x)} = \sum_{n=-\infty}^{\infty} \frac{1}{(x+n)^2}$$

Bernoulli numbers

$$\pi\cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{2x}{x^2-n^2} \Rightarrow x\cot x = 1 + 2\sum_{n=1}^{\infty}\frac{x^2}{x^2-n^2\pi^2} = 1 - 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^{2k}}{n^{2k}\pi^{2k}}$$

Definition

The Bernoulli numbers B_n are defined by the exponential generating function

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!} = 1 - \frac{t}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{t^{2k}}{(2k)!}.$$

$$x\cot x = ix\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}} = ix + \frac{2ix}{e^{2ix}-1} = 1 - \sum_{k=1}^{\infty}(-1)^{k+1}B_{2k}\frac{2^{2k}x^{2k}}{(2k)!}$$

Theorem

For positive integer k,

$$\zeta(2k) = \sum_{n=1}^{\infty}\frac{1}{n^{2k}} = \frac{(-1)^{k+1}2^{2k-1}}{(2k)!}B_{2k}\pi^{2k}.$$

Zeta functions

Definition

The Hurwitz zeta function is defined, for $x > 0$, by

$$\zeta(x, s) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}.$$

And the Riemann zeta function is defined, for $Re(s) > 1$, by

$$\zeta(s) = \zeta(1, s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Recall that

$$\frac{\mathrm{d}^2 \log \Gamma(x)}{\mathrm{d}x^2} = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2} = \zeta(x, 2)$$

Formulas

Definition

$\psi(x) := \Gamma'/\Gamma = \mathrm{d}\left(\log \Gamma(x)\right)/\mathrm{d}x$. $(\Gamma(x+1) = x\Gamma(x) \Rightarrow \psi(x+1) = \psi(x) + 1/x)$

Theorem (Thm. 1.2.7, 1.3.4, 1.6.1, 1.6.2, 1.6.3)

- $\psi(x+n) = \frac{1}{x} + \frac{1}{x+1} + \cdots + \frac{1}{x+n-1} + \psi(x)$, $n = 1, 2, 3, \ldots$
- $\psi\left(\frac{p}{q}\right) = -\gamma - \frac{\pi}{2}\cot\frac{\pi p}{q} - \log q + 2\sum_{n=1}^{\lfloor \frac{q}{2} \rfloor}{}' \cos\frac{2\pi np}{q}\log\left(2\sin\frac{\pi n}{q}\right)$, $0 < p < q$;
  $\sum'$: *if* $q/2 \in \mathbb{N}$, *the last term is divided by* 2.
- $\left(\frac{\partial\zeta(x,s)}{\partial s}\right)_{s=0} = \log\left(\frac{\Gamma(s)}{\sqrt{2\pi}}\right)$
- $\psi(x) = \int_0^\infty \frac{1}{z}\left(e^{-z} - \frac{1}{(1+z)^x}\right)\mathrm{d}z = \int_0^\infty \left(\frac{e^{-z}}{z} - \frac{e^{-xz}}{1-e^{-z}}\right)\mathrm{d}z$
- $$\begin{aligned}\log\Gamma(x) &= \int_0^\infty \left((x-1)e^{-t} - \frac{(1+t)^{-1}-(1+t)^{-x}}{\log(1+t)}\right)\frac{\mathrm{d}t}{t} \\ &= \int_0^\infty \left((x-1)e^{-t} - \frac{e^{-t}-e^{-xt}}{1-e^{-t}}\right)\frac{\mathrm{d}t}{t} \\ &= \left(x-\tfrac{1}{2}\right)\log x - x + \tfrac{1}{2}\log 2\pi + \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t-1}\right)\frac{e^{-tx}}{t}\mathrm{d}t \\ &= \left(x-\tfrac{1}{2}\right)\log x - x + \tfrac{1}{2}\log 2\pi + 2\int_0^\infty \frac{\tan^{-1}\left(\frac{t}{x}\right)}{e^{2\pi t}-1}\mathrm{d}t\end{aligned}$$

Riemann Zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad Re(s) > 0$

- $\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} \mathrm{d}x \Leftarrow \Gamma(s) n^{-s} = \int_0^\infty x^{s-1} e^{-nx} \mathrm{d}x \ (y = nx)$
- $\zeta(s)$ is not the series above, but the analytic continuation of the series. The only simple pole of $\zeta(s)$ is at $s = 1$.

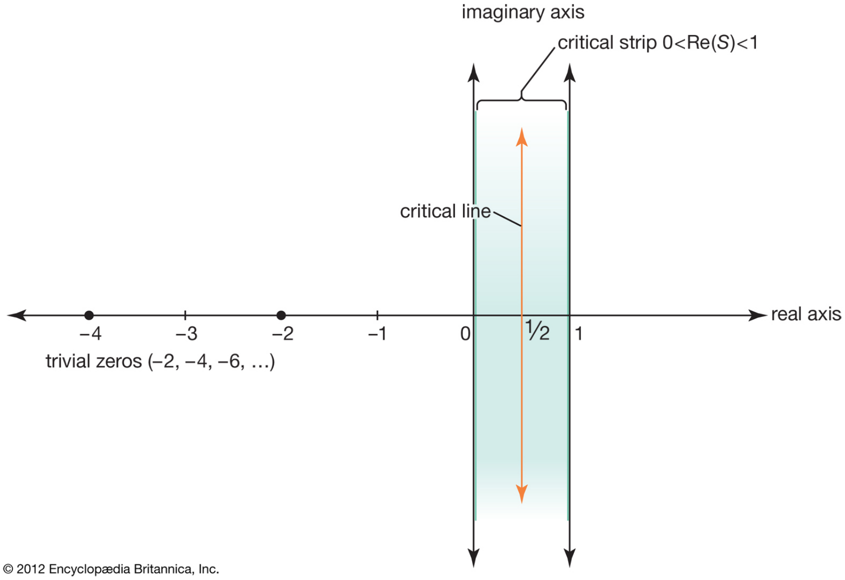
$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} \mathrm{d}x \Rightarrow Res(\zeta, 1) = 1.$$

- $$\begin{aligned} \xi(s) := \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) \quad &\Rightarrow \quad \xi(s) = \xi(1-s) \quad \text{(Thm. 1.3.1)} \\ &\Rightarrow \quad \zeta(-2n) = 0, \ldots n = 1, 2, \ldots \text{(Corol. 1.3.2)} \\ &\qquad\ \ \zeta'(0) = -\frac{1}{2}\log(2\pi) \ldots \text{(Corol. 1.3.3)} \end{aligned}$$

$$\zeta(-n) = \frac{(-1)^n B_{n+1}}{n+1}, \quad n = 0, 1, \ldots$$

$$\text{“}1 + 2 + 3 + \cdots = -\frac{1}{12}.\text{”}$$

RH


imaginary axis
critical strip 0<Re(S)<1
critical line
real axis
−4
−3
−2
−1
0
½
1
trivial zeros (–2, –4, –6, ...)
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Asymptotics

- Stirling (Thm. 1.4.1)

$$\Gamma(x) \sim \sqrt{2\pi} x^{x-\frac{1}{2}} e^{-x} \quad \text{as} \quad x \to \infty$$

$n! \sim \sqrt{2\pi n}(n/e)^n$

- (Thm. 1.4.2) For $x \in \mathbb{C} \backslash (\{0\} \cup \mathbb{R}_-)$,

$$\log \Gamma(x) = \frac{\log(2\pi)}{2} + \left(x - \frac{1}{2}\right) \log x - x + \sum_{j=1}^{m} \frac{B_{2j}}{(2j-1)2j} \cdot \frac{1}{x^{2j-1}} - \underbrace{\frac{1}{2m} \int_0^\infty \frac{B_{2m}(\{t\})}{(x+t)^{2m}} \mathrm{d}t}_{O(x^{-2m+1})}.$$

- (Corol. 1.4.5) For $|\arg x| \le \pi - \delta$, $\delta > 0$,

$$\psi(x) = \log x - \frac{1}{2x} - \sum_{j=1}^{m} \frac{B_{2j}}{2j} \cdot \frac{1}{x^{2j}} + O\left(\frac{1}{x^{2m}}\right)$$

- Euler-Maclaurin summation formula (Appendix D)

$$\sum_{j=a}^{n} f(j) = \int_a^n f(x)\mathrm{d}x + \frac{f(a)+f(n)}{2} + \sum_{j=1}^{m-1} \frac{B_{2j}}{(2j)!} \left(f^{(2j-1)}(n) - f^{(2j-1)}(a)\right)$$

$$+ \int_a^n \frac{B_{2m} - B_{2m}(\{x\})}{(2m)!} f^{(2m)}(x)\mathrm{d}x$$

Hölder's Theorem

Theorem

For every nonnegative integer n, there is NO non-zero polynomial $P \in \mathbb{C}[X; Y_0, \ldots, Y_n]$ such that for any $z \in \mathbb{C} \backslash \{0, -1, -2, \ldots\}$,

$$P(z; \Gamma(z), \Gamma'(z), \ldots, \Gamma^{(n)}(z)) = 0.$$

Let $f(z) = \sin z$, $f'(z) = \cos(z)$, and $P(X; Y_0, Y_1) = Y_0^2 + Y_1^2 - 1$. Then,

$$P(z; f, f') \equiv 0.$$

IDEA of the proof: Assume P is such a polynomial with the lowest degree.

$$P(z+1; \Gamma(z+1), \Gamma'(z+1), \ldots \Gamma^{(n)}(z+1)) = P(z+1; z\Gamma(z), z\Gamma'(z)+\Gamma(z), \ldots, z\Gamma^{(n)}(z)+n\Gamma^{(n-1)}(z))$$

$$Q := P(X+1; XY_0, \ldots, XY_n + nY_{n-1}) \Rightarrow Q = R(X)P$$

$z = m \in \mathbb{N}$ leads to a contradiction.

Generalization

1. q-gamma function (§10.3):

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \Rightarrow [n]_q! = \frac{(q;q)_n}{(1-q)^n}$$

$$\lim_{q\to 1}\frac{(q;q)_n}{(1-q)^n} = \lim_{q\to 1}\prod_{k=0}^{n-1}\frac{1-q^{k+1}}{1-q} = n!$$

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(1-q)^{z-1}(q^z;q)_\infty} \Rightarrow \Gamma_q(n+1) = [n]_q!.$$

2. p-adic gamma function: $|0|_p = 0$ and if $x \neq 0$,

$$|x|_p = 1/p^{\mathrm{ord}_p x}.$$

For $a \in \mathbb{Z}$, $\mathrm{ord}_p a$:=highest power of p that divides a. For $x = a/b \in \mathbb{Q}$, $\mathrm{ord}_p x := \mathrm{ord}_p a - \mathrm{ord}_p b$.

$$f(n) := (-1)^n \prod_{k=1,k\neq p}^{n} k \Rightarrow \Gamma_p(x) := -f(x-1).$$

Poly-

Definition

Recall that $\psi = \mathrm{d}(\log \Gamma(x))/\mathrm{d}x$. Then, $\psi^{(m)}$ is called the the polygamma function of order m.

Definition

The polylogarithm function is defined by a power series in z, which is also a Dirichlet series in s:

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots .$$

Recall that

$$-\log(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k} = \mathrm{Li}_1(z).$$

$$\mathrm{Li}_0(z) = \sum_{k=1}^{\infty} z^k = \frac{1}{1-z} - 1 = \frac{z}{1-z}.$$

Kummer's Fourier Expansion of $\log \Gamma(x)$

Theorem (Thm. 1.7.1)

For $0 < x < 1$

$$\log \frac{\Gamma(x)}{\sqrt{2\pi}} = \frac{-\log(2\sin(\pi x)) + (\gamma + \log(2\pi))(1-2x)}{2} + \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{\log k}{k} \sin(2\pi k x).$$

Eq. (1.7.4)

$$\zeta(x,s) = \frac{2\Gamma(1-x)}{(2\pi)^{1-s}} \left\{ \sin\frac{\pi s}{2} \sum_{m=1}^{\infty} \frac{\cos(2m\pi x)}{m^{1-s}} + \cos\frac{\pi s}{2} \sum_{m=1}^{\infty} \frac{\sin(2m\pi x)}{m^{1-s}} \right\}.$$

Integrals of Dirichlet

Theorem (Thm. 1.8.1)

If

$$V := \left\{ (x_1, \dots, x_n) : x_i \geq 0, \sum_{i=1}^{m} x_i \leq 1, i = 1, \dots, n \right\},$$

then for $Re\alpha_i > 0$, $i = 1, \dots, n$,

$$\int_V x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} \cdots x_n^{\alpha_n - 1} \mathrm{d}x_1 \cdots \mathrm{d}x_n = \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_n)}{\Gamma(1 + \alpha_1 + \cdots + \alpha_n)}.$$

- $$\int_{\mathbb{R}^m} f\left(x_1^2 + \cdots + x_m^2\right) \mathrm{d}x_1 \cdots \mathrm{d}x_m = \frac{2\pi^{\frac{m}{2}}}{\Gamma(m/2)} \int_0^{\infty} r^{m-1} f(r^2) \mathrm{d}r.$$

It can be verified by the method of brackets.

§1.10 Gauss and Jacobi sums

For a prime p, consider the finite field $\mathbb{Z}_p$.

- There are p different homomorphisms: $j = 0, \ldots, p-1$,

$$\begin{aligned} \Psi_j : \mathbb{Z}_p &\to \mathbb{C}^* (\text{multiplicative group of nonzero complex numbers}) \\ x &\mapsto \exp\left(2\pi\sqrt{-1}jx/p\right) \end{aligned}$$

  which are called the additive characters. $(\Psi_j(x)\Psi_j(y) = \Psi_j(x+y))$

- The multiplicative characters are the $p-1$ characters defined similarly as

$$\mathbb{Z}_p^* = \mathbb{Z}_p \backslash \{0\} \cong \mathbb{Z}_{p-1} \to \mathbb{C}^*.$$

Definition

For an additive character Ψ_j and multiplicative character χ_i, the Gauss sums $g_j(\chi_i)$, for $j = 0, \ldots, p-1$ are defined by

$$g_j(\chi_i) = \sum_{x=0}^{p-1} \chi_i(x)\Psi_j(x), \quad (\chi_i(0) := 0, g := g_1).$$

For two multiplicative characters χ and η, the Jacobi sum is defined by

$$J(\chi, \eta) := \sum_{x+y=1} \chi(x)\eta(y).$$

1.10 Gauss and Jacobi sums (Cont'd)

Theorem (Eq. (1.10.12))

If $\chi\eta \neq e$, then

$$J(\chi, \eta) = \frac{g(\chi)g(\eta)}{g(\chi\eta)}.$$

e is the trivial/identity character.