Hypergeometric Functions

January 27th

The hypergeometric series

Definition (The hypergeometric series)

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \cdot \frac{x^n}{n!}, $$

where $(a)_n = a(a+1)\cdots(a+n-1)$. $\boxed{{}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x)}$

Examples (P. 64)

$$ \log(1+x) = x\,{}_2F_1\left(\begin{smallmatrix} 1,1 \\ 2 \end{smallmatrix}; -x\right) \qquad (1-x)^{-a} = {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}; x\right) $$

$$ e^x = {}_0F_0\left(\begin{matrix} - \\ - \end{matrix}; x\right) \qquad \bullet \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{e^t - 1} = \frac{1}{{}_1F_1\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}; t\right)} $$

$$ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = {}_0F_0\left(\begin{matrix} - \\ - \end{matrix}; x\right) $$

The hypergeometric series

Definition (The hypergeometric series)

$$ {}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix};x\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{x^n}{n!}, $$

where $(a)_n = a(a+1)\cdots(a+n-1)$. $\boxed{{}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x)}$

Examples (P. 64)

$$ \log(1+x) = x\,{}_2F_1\left(\begin{smallmatrix} 1,1 \\ 2 \end{smallmatrix};-x\right) \qquad (1-x)^{-a} = {}_1F_0\left(\begin{matrix} a \\ - \end{matrix};x\right) $$

$$ e^x = {}_0F_0\left(\begin{matrix} - \\ - \end{matrix};x\right) \qquad \bullet \sum_{n=0}^{\infty} B_n\frac{t^n}{n!} = \frac{t}{e^t-1} = \frac{1}{{}_1F_1\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix};t\right)} $$

$$ e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = {}_0F_0\left(\begin{matrix} - \\ - \end{matrix};x\right) = \sum_{n=0}^{\infty}\frac{(a)_n}{(a)_n}\cdot\frac{x^n}{n!} = {}_1F_1\left(\begin{matrix} a \\ a \end{matrix};x\right) $$

Convergence

Theorem (Thm. 2.1.1, 2.1.2)

1.

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right) \begin{cases} \text{converges absolutely for all } x, & \text{if } p \leq q; \\ \text{converges absolutely for all } |x| < 1, & \text{if } p = q+1; \\ \text{diverges for all } x \neq 0, & \text{if } \boxed{\begin{matrix} p > q+1 \text{ and the series} \\ \textbf{does not terminate} \end{matrix}} \end{cases} $$

2. For $|x| = 1$

$$ {}_{q+1}F_q\left(\begin{matrix} a_1, \ldots, a_{q+1} \\ b_1, \ldots, b_q \end{matrix}; x\right) \begin{cases} \text{converges absolutely,} & \text{if } Re(\sum b_i - \sum a_i) > 0; \\ \text{converges conditionally,} & \text{if } x \neq 1 \text{ and } Re(\sum b_i - \sum a_i) \in (-1, 0]; \\ \text{diverges,} & \text{if } Re(\sum b_i - \sum a_i) \leq -1. \end{cases} $$

Example

The Wilson polynomials are

$$ W_n(x; a, b, c, d) = \frac{(a+b)_n(a+c)_n(a+d)_n}{a^n}\, {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a+i\sqrt{x}, a-i\sqrt{x} \\ a+b, a+c, a+d \end{matrix}; 1\right) $$

Sometimes, we can write

$$ \sum_{k=0}^{n} x^k = \frac{1-x^{n-1}}{1-x} = {}_1F_1\left(\begin{matrix} -n \\ -n \end{matrix}; x\right) $$

2F1 (Gauss)

Definition

The (Gauss) hypergeometric function ${}_2F_1$ is defined by

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{x^n}{n!} = \left({}_2F_1\left(\begin{matrix} b, a \\ c \end{matrix}; x\right)\right) $$

for $|x| < 1$, and by analytic continuation elsewhere.

Theorem (Thm. 2.2.1, 2.2.2, 2.2.4, 2.2.5)

1. $\boldsymbol{Re}(c) > \boldsymbol{Re}(b) > 0$: ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}\,\mathrm{d}t.$
2. $\boldsymbol{Re}(c-a-b) > 0$: $\sum\limits_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} = {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$
3. $\boldsymbol{Re}(c) > \boldsymbol{Re}(d) > 0$, $x \neq 1$, $|\arg(1-x)| < \pi$: ${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = \frac{\Gamma(c)}{\Gamma(d)\Gamma(c-d)} \int_0^t t^{d-1}(1-t)^{c-d-1}\,{}_2F_1\left(\begin{matrix} a, b \\ d \end{matrix}; xt\right)\mathrm{d}t.$
4. $\boxed{{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = (1-x)^{-a}\,{}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix}; \frac{x}{x-1}\right)} = (1-x)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix} | x\right)$

Proof.

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) \overset{t=1-s}{=} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-s)^{b-1} s^{c-b-1} (1-s+xs)^{-a}\mathrm{d}s $$

$$ = \frac{(1-x)^{-a}\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-s)^{b-1} s^{c-b-1} \left(1 - \frac{xs}{x-1}\right)^{-a} \mathrm{d}s $$

ODE §2.3

${}_2F_1(a, b; c; x)$ satisfies

$$x(1-x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + [c-(a+b+1)x]\frac{\mathrm{d}y}{\mathrm{d}x} - aby = 0,$$

which as three singularities 0, 1, and ∞. Let $t = 1/x$ to get

$$t^2(t-1)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + ((2-c)t + (a+b-1))\, t\frac{\mathrm{d}y}{\mathrm{d}t} - aby = 0.$$

Theorem (Thm. 2.3.1)

A differential equation with three singular points α, β, γ and exponents a_1, a_2; b_1, b_2; and c_1, c_2 respectively, such that $a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = 1$, has the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{1-a_1-a_2}{x-\alpha} + \frac{1-b_1-b_2}{x-\beta} + \frac{1-c_1-c_2}{x-\gamma}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$+ \frac{y}{(x-\alpha)(x-\beta)(x-\gamma)}\left(\frac{(\alpha-\beta)(\alpha-\gamma)a_1 a_2}{x-\alpha} + \frac{(\beta-\alpha)(\beta-\gamma)b_1 b_2}{x-\beta} + \frac{(\gamma-\alpha)(\gamma-\beta)c_1 c_2}{x-\gamma}\right) = 0$$

$$(\alpha, \beta, \gamma) = (0, 1, \infty) \quad (a_1, a_2; b_1, b_2; c_1, c_2) = (0, 1-c; 0, c-a-b; a, b)\,.$$

Terminology

Definition

For a second order ODE $f''(x) + p(x)f'(x) + q(x)f(x) = 0$, point α is

- an ordinary point if both p and q are analytic at $x = \alpha$;
- a regular singular point if p has a pole of order ≤ 1 and q has a pole of order ≤ 2 at $x = \alpha$;
- an irregular singular point otherwise.

Suppose α is a regular singular point. Locally near $x = \alpha$, the ODE has two linearly independent solutions, of the form $f(x) = (x - \alpha)^s g(x)$ for some locally holomorphic function g with $g(\alpha) \neq 0$. s is called the exponent.

Recall the linear fractional transformation: for $p, q, r, s \in \mathbb{C}$ such that $ps - qr \neq 0$, i.e., $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{C})$, define

$$\begin{aligned} f : \bar{\mathbb{C}} &\to \bar{\mathbb{C}} \\ z &\mapsto \frac{pz + q}{rz + s} \end{aligned}$$

$f(z) = \frac{z}{z-1}$, i.e., $(p,q,r,s) = (1,0,1,-1)$ $\boxed{(0,1,\infty) \xrightarrow{f} (0,\infty,1)}$

$$x^\lambda (1-x)^\mu P \begin{Bmatrix} 0 & 1 & \infty & \\ a_1 & b_1 & c_1 & x \\ a_2 & b_2 & c_2 & \end{Bmatrix} = P \begin{Bmatrix} 0 & 1 & \infty & \\ a_1+\lambda & b_1+\mu & c_1-\lambda-\mu & x \\ a_2+\lambda & b_2+\mu & c_2-\lambda-\mu & \end{Bmatrix} \quad (2.3.7)$$

$$P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & x \\ 1-c & c-a-b & b & \end{Bmatrix}$$

$f(z)=\frac{z}{z-1}$, i.e., $(p,q,r,s)=(1,0,1,-1)$ $\boxed{(0,1,\infty)\xrightarrow{f}(0,\infty,1)}$

$$x^{\lambda}(1-x)^{\mu}P\begin{Bmatrix} 0 & 1 & \infty & \\ a_1 & b_1 & c_1 & x \\ a_2 & b_2 & c_2 & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ a_1+\lambda & b_1+\mu & c_1-\lambda-\mu & x \\ a_2+\lambda & b_2+\mu & c_2-\lambda-\mu & \end{Bmatrix} \quad (2.3.7)$$

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & x \\ 1-c & c-a-b & b & \end{Bmatrix} \quad \xrightarrow{f}$$

$$x^\lambda(1-x)^\mu P\begin{Bmatrix} 0 & 1 & \infty & \\ a_1 & b_1 & c_1 & x \\ a_2 & b_2 & c_2 & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ a_1+\lambda & b_1+\mu & c_1-\lambda-\mu & x \\ a_2+\lambda & b_2+\mu & c_2-\lambda-\mu & \end{Bmatrix} \quad (2.3.7)$$

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & x \\ 1-c & c-a-b & b & \end{Bmatrix} \xrightarrow{f} P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & 0 & a & \dfrac{x}{x-1} \\ 1-c & c-a-b & b & \end{Bmatrix}$$

$f(z) = \frac{z}{z-1}$, i.e., $(p,q,r,s) = (1,0,1,-1)$ $\boxed{(0,1,\infty) \xrightarrow{f} (0,\infty,1)}$

$$x^\lambda(1-x)^\mu P \begin{Bmatrix} 0 & 1 & \infty & \\ a_1 & b_1 & c_1 & x \\ a_2 & b_2 & c_2 & \end{Bmatrix} = P \begin{Bmatrix} 0 & 1 & \infty & \\ a_1+\lambda & b_1+\mu & c_1-\lambda-\mu & x \\ a_2+\lambda & b_2+\mu & c_2-\lambda-\mu & \end{Bmatrix} \quad (2.3.7)$$

$$\begin{aligned} P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & x \\ 1-c & c-a-b & b & \end{Bmatrix} \quad &\xrightarrow{f} \quad P \begin{Bmatrix} 0 & \infty & 1 & \\ 0 & 0 & a & \dfrac{x}{x-1} \\ 1-c & c-a-b & b & \end{Bmatrix} \\ &= \quad P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & a & 0 & \dfrac{x}{x-1} \\ 1-c & b & c-a-b & \end{Bmatrix} \end{aligned}$$

$f(z) = \frac{z}{z-1}$, i.e., $(p,q,r,s) = (1,0,1,-1)$ $\boxed{(0,1,\infty) \xrightarrow{f} (0,\infty,1)}$

$$x^{\lambda}(1-x)^{\mu} P\begin{Bmatrix} 0 & 1 & \infty & \\ a_1 & b_1 & c_1 & x \\ a_2 & b_2 & c_2 & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ a_1+\lambda & b_1+\mu & c_1-\lambda-\mu & x \\ a_2+\lambda & b_2+\mu & c_2-\lambda-\mu & \end{Bmatrix} \quad (2.3.7)$$

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & x \\ 1-c & c-a-b & b & \end{Bmatrix} \xrightarrow{f} P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & 0 & a & \frac{x}{x-1} \\ 1-c & c-a-b & b & \end{Bmatrix}$$

$$= P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & a & 0 & \frac{x}{x-1} \\ 1-c & b & c-a-b & \end{Bmatrix}$$

$$= \left(1-\frac{x}{x-1}\right)^{a} P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & \frac{x}{x-1} \\ 1-c & b-a & c-b & \end{Bmatrix}$$

$f(z) = \frac{z}{z-1}$, i.e., $(p, q, r, s) = (1, 0, 1, -1)$ $\boxed{(0,1,\infty) \xrightarrow{f} (0,\infty,1)}$

$$x^{\lambda}(1-x)^{\mu} P \begin{Bmatrix} 0 & 1 & \infty & \\ a_1 & b_1 & c_1 & x \\ a_2 & b_2 & c_2 & \end{Bmatrix} = P \begin{Bmatrix} 0 & 1 & \infty & \\ a_1+\lambda & b_1+\mu & c_1-\lambda-\mu & x \\ a_2+\lambda & b_2+\mu & c_2-\lambda-\mu & \end{Bmatrix} \quad (2.3.7)$$

$$\begin{aligned} P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & x \\ 1-c & c-a-b & b & \end{Bmatrix} &\xrightarrow{f} P \begin{Bmatrix} 0 & \infty & 1 & \\ 0 & 0 & a & \frac{x}{x-1} \\ 1-c & c-a-b & b & \end{Bmatrix} \\ &= P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & \color{red}{a} & 0 & \frac{x}{x-1} \\ 1-c & b & c-a-b & \end{Bmatrix} \\ &= \left(1 - \frac{x}{x-1}\right)^{a} P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & \frac{x}{x-1} \\ 1-c & b-a & c-b & \end{Bmatrix} \\ &= (1-x)^{-a} P \begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & \frac{x}{x-1} \\ 1-c & b-a & c-b & \end{Bmatrix} \end{aligned}$$

$${}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = (1-x)^{-a} {}_2F_1\left(\begin{matrix} a, c-b \\ c \end{matrix}; x\right)$$

Exercise: Try one of the transformations in (2.3.9).

Quadratic §3.9

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & c & a & x \\ \frac{1}{2} & d & b & \end{Bmatrix} \overset{x=t^2}{=\!=\!=\!=} P\begin{Bmatrix} -1 & 1 & \infty & \\ c & c & 2a & t \\ d & d & 2b & \end{Bmatrix} \Longleftarrow \text{Manipulating the ODE}$$

Theorem (Thm. 3.9.1)

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & c & a & t^2 \\ \frac{1}{2} & d & b & \end{Bmatrix} = P\begin{Bmatrix} -1 & 1 & \infty & \\ c & c & 2a & t \\ d & d & 2b & \end{Bmatrix} = P\begin{Bmatrix} 0 & 1 & \infty & \\ c & c & 2a & \frac{t+1}{2} \\ d & d & 2b & \end{Bmatrix}$$

Example

$$P\begin{Bmatrix} 1 & 0 & \infty & \\ 0 & c & a & 1-t^2 \\ \frac{1}{2} & d & b & \end{Bmatrix} = P\begin{Bmatrix} 0 & \infty & 1 & \\ c & c & 2a & \frac{t+1}{t-1} \\ d & d & 2b & \end{Bmatrix}$$

$$(a,b,c,d) = \left(\frac{a}{2}, \frac{1-2b+a}{2}, 0, b-a\right) \text{ and } (t+1)/(t-1) = x$$

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & 0 & a & x \\ b-a & b-a & 1-2b+a & \end{Bmatrix} = P\begin{Bmatrix} 1 & 0 & \infty & \\ 0 & 0 & \frac{a}{2} & -\frac{4x}{(1-x)^2} \\ \frac{1}{2} & b-a & \frac{1-2b+a}{2} & \end{Bmatrix} \text{(Thm. 3.1.1)}$$

Contiguous relations

$$\frac{\mathrm{d}}{\mathrm{d}x}\left({}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right)\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{x^n}{n!}\right)=\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}}\cdot\frac{x^n}{n!}=\frac{ab}{c}\,{}_2F_1\left(\begin{matrix}a+1,b+1\\c+1\end{matrix};x\right)$$

$$\text{ODE}\Rightarrow {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right)=(1-x)\,{}_2F_1\left(\begin{matrix}a+1,b\\c\end{matrix};x\right)+\frac{(c-b)x}{c}\,{}_2F_1\left(\begin{matrix}a+1,b\\c+1\end{matrix};x\right)$$

$$F:={}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right)\Longrightarrow\begin{cases}x\frac{\mathrm{d}F}{\mathrm{d}x}=a(F(a+)-F)=b(F(b+)-F)=(c-1)(F(c-)-F)\\ x(1-x)\frac{\mathrm{d}F}{\mathrm{d}x}=(c-a)F(a-)+(a-c+bx)F\\ \qquad\qquad\qquad=(c-b)F(b-)+(b-c+ax)F\\ c(1-x)\frac{\mathrm{d}F}{\mathrm{d}x}=(c-a)(c-b)F(c+)+c(a+b-c)F\end{cases}$$

Example (2.5.11)

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix};x\right)={}_2F_1\left(\begin{matrix}a,b+1\\c+1\end{matrix};x\right)-\frac{a(c-b)}{c(c+1)}x\,{}_2F_1\left(\begin{matrix}a+1,b+1\\c+2\end{matrix};x\right)$$

Continued fractions §2.5

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; x\right) = {}_2F_1\left(\begin{matrix} a, b+1 \\ c+1 \end{matrix} ; x\right) - \frac{a(c-b)}{c(c+1)} x \, {}_2F_1\left(\begin{matrix} a+1, b+1 \\ c+2 \end{matrix} ; x\right) $$

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; x\right) = (1-x) \, {}_2F_1\left(\begin{matrix} a+1, b \\ c \end{matrix} ; x\right) + \frac{(c-b)x}{c} \, {}_2F_1\left(\begin{matrix} a+1, b \\ c+1 \end{matrix} ; x\right) $$

$$ \frac{{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} ; x\right)}{{}_2F_1\left(\begin{matrix} a, b+1 \\ c+1 \end{matrix} ; x\right)} = 1 - \frac{a(c-b)}{c(c+1)} x \cdot \frac{1}{\frac{{}_2F_1\left(\begin{matrix} a, b+1 \\ c+1 \end{matrix} ; x\right)}{{}_2F_1\left(\begin{matrix} a+1, b+1 \\ c+2 \end{matrix} ; x\right)}} $$

Continued fractions §2.5

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = {}_2F_1\left(\begin{matrix} a, b+1 \\ c+1 \end{matrix}; x\right) - \frac{a(c-b)}{c(c+1)} x\, {}_2F_1\left(\begin{matrix} a+1, b+1 \\ c+2 \end{matrix}; x\right) $$

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right) = (1-x)\, {}_2F_1\left(\begin{matrix} a+1, b \\ c \end{matrix}; x\right) + \frac{(c-b)x}{c}\, {}_2F_1\left(\begin{matrix} a+1, b \\ c+1 \end{matrix}; x\right) $$

$$ \frac{{}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix}; x\right)}{{}_2F_1\left(\begin{matrix} a, b+1 \\ c+1 \end{matrix}; x\right)} = 1 - \frac{a(c-b)}{c(c+1)} x \cdot \frac{1}{\dfrac{{}_2F_1\left(\begin{matrix} a, b+1 \\ c+1 \end{matrix}; x\right)}{{}_2F_1\left(\begin{matrix} a+1, b+1 \\ c+2 \end{matrix}; x\right)}} $$

$$ = 1 - \frac{\frac{a(c-b)}{c(c+1)} x}{\frac{c-b}{c+1} x + \dfrac{1-x}{\dfrac{{}_2F_1\left(\begin{matrix} a+1, b \\ c+1 \end{matrix}; x\right)}{{}_2F_1\left(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}; x\right)}}} $$

$$ = \cdots $$

Generalized ${}_pF_q$

$$ {}_{p+1}F_{q+1}\left(\begin{matrix} a_1,\dots,a_p,c \\ b_1,\dots,b_q,d \end{matrix};x\right) = \frac{\Gamma(d)}{\Gamma(c)\Gamma(d-c)}\int_0^1 t^{c-1}(1-t)^{d-c-1}\,{}_pF_q\left(\begin{matrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{matrix};tx\right)\mathrm{d}t $$

Definition

A series ${}_{q+1}F_q\left(\begin{matrix} a_1,\dots,a_q,a_{q+1} \\ b_1,\dots,b_q \end{matrix};x\right)$ is called balanced if $x=1$, one of the numerator parameters is a negative integer, and $a_1+\cdots+a_q+a_{q+1}+1=b_1+\cdots+b_q$.

Examples

$$ {}_3F_2\left(\begin{matrix} -n,-a,-b \\ c,1-a-b-n-c \end{matrix};1\right) = \frac{(c+a)_n(c+b)_n}{(c)_n(c+a+b)_n} $$

and more as (2.2.9). See also §3.3

Evaluations

$$ {}_3F_2\left(\begin{matrix} a, -b, -c \\ a+b+1, a+c+1 \end{matrix} ;1\right) = \frac{\Gamma\left(\frac{a}{2}+1\right)\Gamma(a+b+1)\Gamma(a+c+1)\Gamma\left(\frac{a}{2}+b+c+1\right)}{\Gamma(a+1)\Gamma\left(\frac{a}{2}+b+1\right)\Gamma\left(\frac{a}{2}+c+1\right)\Gamma(a+b+c+1)} $$

$$ {}_5F_4\left(\begin{matrix} a, \frac{a}{2}+1, -b, -c, -d \\ \frac{a}{2}, a+b+1, a+c+1, a+d+1 \end{matrix} ;1\right) = \frac{\Gamma(a+b+1)\Gamma(a+c+1)\Gamma(a+d+1)\Gamma(a+b+c+d+1)}{\Gamma(a+1)\Gamma(a+b+c+1)\Gamma(a+b+d+1)\Gamma(a+c+d+1)} $$

$$ {}_2F_1\left(\begin{matrix} a, b \\ a-b+1 \end{matrix} ;-1\right) = \frac{\Gamma(a-b+1)\Gamma((a/2)+1)}{\Gamma(a+1)\Gamma((a/2)-b+1)} $$

$$ {}_3F_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix} ;1\right) \overset{\text{condition}}{==} \frac{\pi^2\Gamma(d)\Gamma(e)[\cos\pi d\cos\pi e+\cos\pi a\cos\pi b\cos\pi c]}{\Gamma(d-a)\Gamma(d-b)\Gamma(d-c)\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)} (P.\ 116) $$

$$ {}_2F_1\left(\begin{matrix} a, b \\ \frac{a+b+1}{2} \end{matrix} ;\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)} \qquad {}_2F_1\left(\begin{matrix} a, 1-a \\ c \end{matrix} ;\frac{1}{2}\right) = \frac{\Gamma\left(\frac{c}{2}\right)\Gamma\left(\frac{c+1}{2}\right)}{\Gamma\left(\frac{c+a}{2}\right)\Gamma\left(\frac{c-a+1}{2}\right)} $$

1.
$$
{}_7F_6\left(\begin{matrix} a, \frac{a}{2}+1, b, c, d, e, f \\ \frac{a}{2}, a-b+1, a-c+1, a-d+1, a-e+1, a-f+1 \end{matrix}; 1\right)
= \frac{\Gamma(a-d+1)\Gamma(a-e+1)\Gamma(a-f+1)\Gamma(a-d-e-f+1)}{\Gamma(a+1)\Gamma(a-e-f+1)\Gamma(a-d-e+1)\Gamma(a-d-f+1)}
\times {}_4F_3\left(\begin{matrix} a-b-c+1, d, e, f \\ a-b+1, a-c+1, d+e+f-a \end{matrix}\right)
$$

2. Let $F := {}_3F_2\left(\begin{matrix} a, b, d \\ c, e \end{matrix}; x\right)$ and $F_+ = {}_3F_2\left(\begin{matrix} a+1, b+1, d+1 \\ c+1, e+1 \end{matrix}; x\right)$

$$
F(a-, e-) - F = \frac{(a-e)bcd}{(e-1)efg} F_+(a-)
$$

$$
F(a+, e+) - F = \frac{(e-a)bcd}{e(e+1)fg} F_+(e+)
$$

$$
F(e+, f-) - F = \frac{(e-f+1)abcd}{e(e+1)fg} F_+(e+)
$$

Related special functions

1. The Jacobi polynomial of degree n is defined by

$$P_n^{(\alpha,\beta)}(x) := \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right)$$

   Contiguous relations of ${}_2F_1$ (2.5.15) yields the three-term recurrence of $P_n^{(\alpha,\beta)}(x)$.

2. Wilson polynomial

$$W_n(x; a, b, c, d) = \frac{(a+b)_n(a+c)_n(a+d)_n}{a^n}\,{}_4F_3\left(\begin{matrix}-n, n+a+b+c+d-1, a+i\sqrt{x}, a-i\sqrt{x}\\ a+b, a+c, a+d\end{matrix}; 1\right)$$

   or alternatively

$$p_n(x^2) = a^n W_n(x^2; a, b, c, d).$$

Dilogarithms §2.6 $\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2}$

$$\mathrm{Li}_2(x) = -\int_0^x \frac{\log(1-t)}{t}\mathrm{d}t = \int_0^x {}_2F_1\left(\begin{matrix}1,1\\2\end{matrix};t\right)dt = x{}_3F_2\left(\begin{matrix}1,1,1\\2,2\end{matrix};x\right)$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\tfrac{x}{x-1}\right) = -\tfrac{\log^2(1-x)}{2} \qquad \mathrm{Li}_2(x) + \mathrm{Li}_2(-x) = \tfrac{\mathrm{Li}_2(x^2)}{2}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \tfrac{\pi^2}{6} - \log x \log(1-x) \qquad \sum_{k=0}^{n-1}\mathrm{Li}_2(\omega^k x) = \tfrac{1}{n}\mathrm{Li}_2(x^n), \text{if } \omega^n = 1.$$

$$\mathrm{Li}_2(0) = 0, \quad \mathrm{Li}_2(1) = \tfrac{\pi^2}{6}, \quad \mathrm{Li}_2(-1) = -\tfrac{\pi^2}{12}, \quad \mathrm{Li}_2\left(\tfrac{1}{2}\right) = \tfrac{\pi^2}{12} - \tfrac{\log^2 2}{2}$$

$$\mathrm{Li}_2\left(\tfrac{3-\sqrt{5}}{2}\right) = \tfrac{\pi^2}{15} - \tfrac{1}{4}\left[\log\left(\tfrac{3-\sqrt{5}}{2}\right)\right]^2, \qquad \mathrm{Li}_2\left(\tfrac{\sqrt{5}-1}{2}\right) = \tfrac{\pi^2}{10} - \left[\log\left(\tfrac{\sqrt{5}-1}{2}\right)\right]^2$$

$$\mathrm{Li}_2\left(\tfrac{1-\sqrt{5}}{2}\right) = -\tfrac{\pi^2}{15} + \tfrac{1}{2}\left[\log\left(\tfrac{\sqrt{5}-1}{2}\right)\right]^2, \quad \mathrm{Li}_2\left(-\tfrac{1+\sqrt{5}}{2}\right) = \tfrac{-\pi^2}{10} + \tfrac{1}{2}\left[\log\left(\tfrac{\sqrt{5}+1}{2}\right)\right]^2$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{Li}_s(x) = \frac{\mathrm{Li}_{s-1}(x)}{x} \Rightarrow \mathrm{Li}_s(x) = x\ {}_{s+1}F_s\left(\begin{matrix}1,1,\ldots,1\\2\ldots,2\end{matrix};t\right)$$

AGM §3.2 (Arithmetic-Geometric Mean)

Definition

Define two sequence $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ as follows. Let $a_0 = a$ and $b_0 = b$, such that $a \geq b > 0$. Recursively, define

$$a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \sqrt{a_n b_n}$$

Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n =: M(a, b)$$

is called the arithmetic-geometric mean of a and b.

Definition

The complete elliptic integral of the first kind is defined by, for $0 < k < 1$

$$K := K(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}}$$

The second kind: $E(k) := \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\mathrm{d}\theta$.

$$K' := K(k'), \quad E' = E(k') \quad k' = \sqrt{1-k^2}$$

Elliptic Integrals

$$K = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} = \frac{\pi}{2}\,{}_2F_1\left(\begin{matrix}\frac{1}{2},\frac{1}{2}\\1\end{matrix};k^2\right) \qquad \frac{1}{M(a,b)} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}$$

$$\frac{1}{M(1,k)} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{\cos^2\theta+k^2\sin^2\theta}} = K'$$

$$EK' + E'K - KK' = \frac{\pi}{2}$$

$$\pi = \frac{M^2(\sqrt{2},1)}{1-\sum_{n=0}^{\infty} 2^n c_n^2}, \qquad c_n^2 = a_n^2 - b_n^2, \text{ with } a_0 = 1, b_0 = \frac{1}{\sqrt{2}}.$$

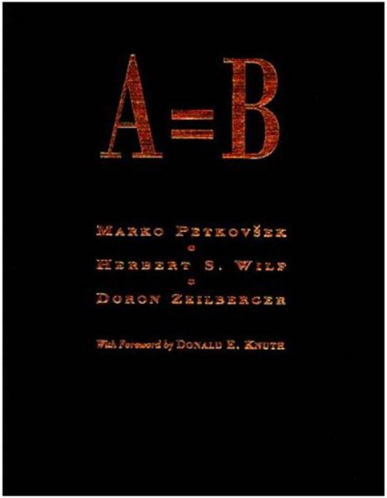
Definition

A Ramanujan's modular equation

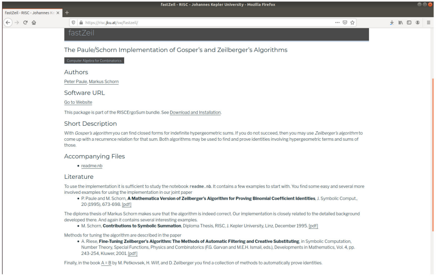
$$n\frac{K'}{K} = \frac{L'}{L}$$

where $L = K(\ell)$ and $L' = K(\ell')$, for $\ell' = \sqrt{1-\ell^2}$.

W-Z Method



https://www.math.upenn.edu/~wilf/AeqB.html



fastZeil - RISC - Johannes Kepler University - Mozilla Firefox

https://risc.jku.at/sw/fastzeil/

fastZeil

The Paule/Schorn Implementation of Gosper's and Zeilberger's Algorithms

Computer Algebra for Combinatorics

Authors

Peter Paule, Markus Schorn

Software URL

Go to Website

This package is part of the RISCErgoSum bundle. See Download and Installation.

Short Description

With *Gosper's algorithm* you can find closed forms for indefinite hypergeometric sums. If you do not succeed, then you may use *Zeilberger's algorithm* to come up with a recurrence relation for that sum. Both algorithms may be used to find and prove identities involving hypergeometric terms and sums of those.

Accompanying Files

- readme.nb

Literature

To use the implementation it is sufficient to study the notebook `readme.nb`. It contains a few examples to start with. You find some easy and several more involved examples for using the implementation in our joint paper

- P. Paule and M. Schorn, **A Mathematica Version of Zeilberger's Algorithm for Proving Binomial Coefficient Identities**, J. Symbolic Comput., 20 (1995), 673-698. [pdf]

The diploma thesis of Markus Schorn makes sure that the algorithm is indeed correct. Our implementation is closely related to the detailed background developed there. And again it contains several interesting examples.

- M. Schorn, **Contributions to Symbolic Summation**, Diploma Thesis, RISC, J. Kepler University, Linz, December 1995. [pdf]

Methods for tuning the algorithm are described in the paper

- A. Riese, **Fine-Tuning Zeilberger's Algorithm: The Methods of Automatic Filtering and Creative Substituting**, in Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics (F.G. Garvan and M.E.H. Ismail, eds.), Developments in Mathematics, Vol. 4, pp. 243-254, Kluwer, 2001. [pdf]

Finally, in the book A = B by M. Petkovsek, H. Wilf, and D. Zeilberger you find a collection of methods to automatically prove identities.

https://risc.jku.at/sw/fastzeil/

Hypergeometric series

Definition

1. A geometric series $\sum_{k=0}^{\infty} t_k$ is one in which the ratio of every two consecutive terms is constant. $t_k = cx^k$ for some constants x and $c = t_0$.
2. A hypergeometric series is one in which $t_0 = 1$ and the ration of two consecutive terms is a rational function of the summation index k.

Problem

Find $f(n) := \sum_{k=A}^{B} F(n,k)$, where $F(n,k)$ is is a hypergeometric term in both arguments. Namely, both $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are rational functions in n and k.

The idea is to find another function $G(n,k)$ such that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k).$$

This shows that $f(n) = G(n,B+1) - G(n,A)$.

Example (3.11.7)

$$ {}_4F_3\left(\begin{matrix} \frac{a}{2}, \frac{a+1}{2}, b+n, -n \\ \frac{b}{2}, \frac{b+1}{2}, a+1 \end{matrix} ; 1\right) = \frac{(b-a)_n}{(b)_n} $$

$$ \sum_{k=0}^{n} \frac{\left(\frac{a}{2}\right)_k \left(\frac{a+1}{2}\right)_k (b+n)_k (-n)_k}{\left(\frac{b}{2}\right)_k \left(\frac{b+1}{2}\right)_k (a+1)_k k!} = \frac{(b-a)_n}{(b)_n} $$

$$ F(n,k) = \frac{\left(\frac{a}{2}\right)_k \left(\frac{a+1}{2}\right)_k (b+n)_k (-n)_k (b)_n}{\left(\frac{b}{2}\right)_k \left(\frac{b+1}{2}\right)_k (a+1)_k k! (b-a)_n} $$